Grade 7/8 Math Circles
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Introduction to Proofs

Introduction

In your math classes so far, you have probably been given some facts that you have been told are true. For example, you may have seen Pythagorean Theorem which states that in a right-angled triangle with side lengths a, b, and c, $a^2 + b^2 = c^2$.

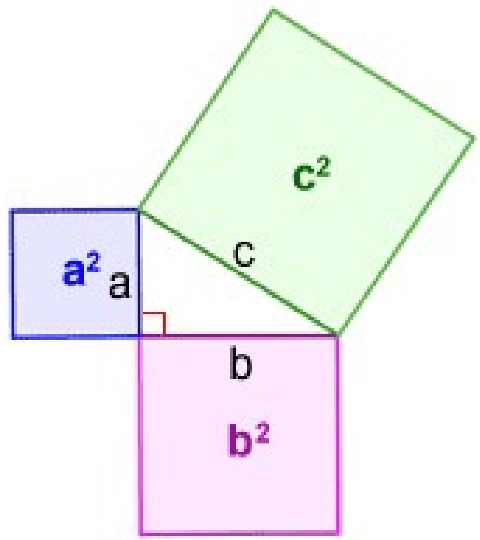

Figure 1: Image retrieved from CalCon Calculators and Converters

If you are given two of the side lengths, you can determine the length of the third side through a little algebra. But why does this always work?

Why does this always work? This question is the main idea behind proofs. A mathematical **proof** contains series of facts that we assume to be true, and when we combine these facts together, we can show the statement is true **for all cases**.

Stop and Think

Why is proving a statement is true **for all cases** useful?

Hypothesis and Conclusion

When proving a statement, you are often given some **hypothesis**. These are statements you can accept as true. The **conclusion** is the statement you want to prove. By letting the hypothesis be true and by using reasoning skills and other facts, you can show the conclusion is true, or can **disprove** the conclusion (more on this later).

Example 1

Does the following reasoning hold true?

Hypotheses:

- If I did homework this evening, then I didn't finish it in class.
- I did homework this evening.

Conclusion: I didn't finish my homework in class.

Solution:

This argument is correct! Since we know I did homework this evening, we can deduce from the first point that I did not finish homework in class.

Exercise 1

Does the following reasoning hold true?

Hypotheses:

- If I did homework this evening, then I didn't finish it in class.
- I finished my homework in class.

Conclusion: I didn't do homework this evening.

Implications

An implication is a sentence where one statement **implies** another. Implications are sometimes called "if, then" statements because they are always in the format, "if A, then B." A is the hypothesis and B is the conclusion. To prove an implication, assume the hypothesis is true and use other facts and reasoning to prove the conclusion is true.

Example 2

Consider the following statement:
"If I did homework this evening, then I did not finish it in class."

This statement is actually an implication!
We like using symbols in math to reduce how much we need to write, so we can let:

- A = "I did homework this evening" and;
- B = "I did not finish it in class."

Further, $\implies$ is the symbol for implications. A $\implies$ B is read "A implies B."

Before we try a more mathematical proof, we need to learn about factoring and distribution.

Factoring and Distribution

Definitions

An **expression** is a mathematical sentence with at least one term.
A **term** is a number, variable, or the product of a number and variables.
An **equation** is a mathematical sentence in which an expression is equal to another expression or number.

Example 3

$3x + 2$ is an expression with 2 terms.
$4a + b = 3$ is an equation with 3 terms.
$6x + 8y - z = 1$ is an equation with 4 terms.

Factoring

Factoring is the process of writing an expression as a product of factors. There are many strategies for factoring expressions, but for today's lesson, we will focus on **common factoring**. This involves finding a number or variable by which each term can be divided.

Exercise 2

Perform the following calculations. Try to find patterns in your answers.

1. $6 + 3 =$

   $3(2 + 1) =$

2. $15 - 10 =$

   $5(3 - 2) =$

3. $4(3) + 4(2) =$

   $4(3 + 2) =$

Notice the similarities between the answers. This is how FACTORING works!

In the first question, since each term can be divided by 3, we can factor out the 3 and $2 + 1$ is placed in brackets. Why? Because $6 \div 3 = 2$ and $3 \div 3 = 1$.

In the second question, both 15 and -10 can be divided by 5 so we *factor out* a 5. $15 \div 5 = 3$ and $-10 \div 5 = -2$, so $3 - 2$ is in the brackets.

Stop and Think

In the third questions from Exercise 2, what did we factor out? Why is $3 + 2$ in the brackets?

Now let's try factoring with variables too. It's really the same process! Find a common factor between all the terms and factor it out.

Exercise 3

Now try factoring these on your own!

1. $4x - 4y + 8$
2. $2a + 6ab + 5ac$
3. $16d + 12de$

Distribution

The **Distributive Property** is a property that helps us undo factoring. It gets its name because you "distribute" the factor that is outside the brackets into the brackets. We do this by multiplying the factor outside the brackets by every term inside the brackets.

Example 4

Use the distributive property to expand the expression $5(x + 2y)$.

Solution:

5 is the factor outside the brackets and there are two terms inside the brackets; x and $2y$. We multiply 5 by each of these terms.

$$5(x+2y) = 5x+10y$$

5x 10y

Exercise 4

Now try expanding these on your own!

1. $x(x + 2)$
2. $3(a + b + 2c)$
3. $2y(x + 4)$

TIP: Whenever you need to factor or expand, you can check your answer by factoring your answer if you expanded or by expanding your answer if you factored. If you did it right, you should get the expression you started with. This is because factoring and expanding undo each other.

Now that we know factoring and distribution, let's try some mathematical proofs. Some of them might seem obviously true when thinking about some examples, but let's actually prove they are true for all cases.

Even, Odd, Divisiblity

This brief section will help with the proofs that follow.

16 is even because 16 is a multiple of 2. That is, $16 = 2(8)$.

19 is odd because 19 is not a mutliple of 2. That is, $19 = 2(9) + 1$.

28 is divisible by 7 (or 28 is a mutliple of 7) because $28 = 7(4)$.

More generally,

- To prove a number x is even, show $x = 2k$ for some integer k.
- To prove a number x is odd, show $y = 2k + 1$ for some integer k.
- To prove a number x is divisible by 8, for example, show $x = 8k$ for some integer k.

Implications Continued

Example 5

If both x and y are odd numbers, then $x + y$ is even.

(a) Write the above statement as an implication with symbols.

(b) Prove the above statement.

Solution:

(a) Let A = "both x and y are odd." Let B = "$x + y$ is even."

Then we can write the statement as A $\implies$ B.

(b) *Proof.* The hypothesis is A, so we assume this is true. We want to show B. We can do this by showing $x + y = 2k$ for some integer k.

Since both x and y are odd, we can write $x = 2a + 1$ and $y = 2b + 1$ for some integers a and b. Why can we do this? Because even numbers are always divisible by 2, so we can write any even number as 2 times another number. Odd numbers are always one away from an even number, so that's where the +1 comes from. Now we consider $x + y$.

$$\begin{aligned} x + y &= 2a + 1 + 2b + 1 \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1) && \text{(common factor a 2)} \\ &= 2k && \text{(where } k = a + b + 1\text{)} \end{aligned}$$

Since $x + y = 2k$ for some integer k, $x + y$ is even!

□

NOTE: The little box at the end of the last example indicates where the proof ends. QED is also used to indicate a proof is complete. QED stands for "quod erat demonstrandum" which translates from Latin to "what was to be shown".

Exercise 5
Prove the following implication.
If x and y are both even, then $x + y$ is even.

Disproving Implications

So far, we've seen statements that have all been true. However, statements can also be false. How do we *prove* it is false? That's when counter-examples are useful.

A **counter-example** is an example such that the hypothesis is true, but the conclusion is false. Let's try another example about finishing homework.

Example 6
Does the following reasoning hold true?
Hypotheses:

- If I did homework this evening, then I didn't finish it in class.
- I didn't do homework this evening.

Conclusion: I finished my homework in class.

Solution:
This argument is false. We only know something about people who did their homework this evening, not about people who didn't do homework this evening. What if I was sick and had no energy to do my homework. Maybe I will do it tomorrow because I had to babysit tonight.

In the above example, a few counter-examples were provided. In the "sick person" example, the hypotheses were true but I didn't even go to class, so I definitely didn't finish the homework in class. Thus, the conclusion is false and we disproved the statement.

NOTE: Multiple counter-examples were provided in the last example, but you only need to find one to disprove an implication!

Here's an example for disproving implications that is more mathematical.

Example 7

If x and y are perfect squares, then $x + y$ is a perfect square.

Note: A perfect square is the product of an integer and itself. For example 25 is a perfect square because $5 \times 5 = 25$.

Solution:

Proof. This is false. We can show this with a counter-example.

Let $x = 16$ and let $y = 49$. These are perfect squares because $4 \times 4 = 16$ and $7 \times 7 = 49$. So, the hypothesis is true.

However, $16 + 49 = 65$ and 65 is not a perfect square because there is no integer k such that $k \times k = 65$.

□

Exercise 6

Disprove the following statement. If x and y are both odd numbers, then $x^2 + 3y$ is odd.

Converse

The converse of an implication is the new implication you make when you reverse the order of the statements. That is, if A $\implies$ B is your original implication, the converse is B $\implies$ A.

Exercise 7

Let A "I scored a goal." Let B = "I get a McDonalds meal."

(a) Write the implication A $\implies$ B in words.

(b) Write the converse of A $\implies$ B in words.

Stop and Think

In the previous exercise, let's say we know A $\implies$ B is true. Does this mean B $\implies$ A is also true?

No! Here's a counter-example. What if I get a McDonalds meal for a different reason. Maybe I did well on a test or I was just *really* hungry and needed something to eat. In these cases, B is true but A could be true or false. We don't know. Now we will try a mathematical exercise of the converse.

Exercise 8

In Exercise 5, you proved that if x and y are both even numbers, then $x + y$ is even. Now let's focus on the converse of this statement.

(a) Write the converse of this statement in words.

(b) If you think the converse is true, prove it!. If you think it is false, give a counter-example!

Example 8

(a) Let x be an integer and consider the following statement.

"If x^2 is even, then x is even."

This statement is true, but proving it requires a proof technique which is more complex (in case you are interested, the technique is called the contrapositive).

For today, just test some examples to convince yourself it is true.

NOTE: Finding these examples does **not** prove the statement is true.

(b) Now consider the converse. "If x is even, then x^2 is even."

If you think this is true, prove it. If you think it is false, find a counter-example.

Solution:

(a) Here are some examples to convince yourself the implication is true.

$x^2 = 16$ is even. Thus, $x = 4$ which is also even.

$x^2 = 100$ is even. Thus, $x = 10$ which is also even.

This is not a proof because we have **not** proved the statement **for all cases**.

(b) *Proof.* Let the hypothesis be true. So, x is even which means we can say $x = 2a$ for some integer a. Consider x^2.

$$\begin{aligned} x^2 &= (x)(x) \\ &= (2a)(2a) \\ &= 4a^2 \\ &= 2(2a^2) && \text{(common factor a 2)} \\ &= 2k && \text{(where } k = 2a^2\text{)} \end{aligned}$$

Since $x^2 = 2k$ for some integer k, x^2 is even!

□

This time the implication and converse are both true! This means sometimes the converse is false and sometimes it is true. When both directions of the statement are true, the statement gets a special name. They are called **If and Only If** statements!

If and Only If Statements

If and only if statements combine an implication and their converse into one logical statement. The symbol for if and only if is $\iff$ and we would read A $\iff$ B as "A if and only if B." This is the same as A $\implies$ B AND B $\implies$ A. "If and only if" is often shortened to "iff".

Example 9
"I will bring an umbrella if and only if the weather says it will rain."
Thinking back to implications, this means:

- If I bring an umbrella, then the weather says it will rain. AND
- If the weather says it will rain, then I bring an umbrella.

If and only if statements are powerful! We know all there is to know about the situation.

Expressing a Number in Expanded Form

95 has the digits 9 and 5. It can be written as $10(9) + 5 = 95$

258 has the digits 2, 5, and 8. It can be written as $100(2) + 10(5) + 8 = 258$

More generally,

- A two-digit number with digits a and b can be written as $10a + b$.
- A three-digit number with digits a, b, and c can be written as $100a + 10b + c$.

Exercise 9
Write the following numbers in expanded form.

(a) 62

(b) 389

Example 10

Prove the divisibility by 3 rule for two-digit numbers. That is, prove the following statement. A two-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Solution:

Proof. Let x be a two-digit number with digits a and b. Then $x = 10a + b$ because a is the tens digit and b is the ones digit.

$\Longrightarrow$ (This implication symbol means we are proving the forward direction. "If x is divisible by 3, then the sum of its digits is divisible by 3.")

Since x is divisible by 3, we can find an integer k such that $x = 3k$. Therefore,

$$\begin{aligned} 10a + b &= 3k && \\ a + b &= 3k - 9a && \text{(subtract } 9a \text{ from both sides)} \\ a + b &= 3(k - 3a) && \text{(common factor a 3)} \\ a + b &= 3m && \text{(where } m = k - 3a\text{)} \end{aligned}$$

Therefore, $a + b$, which is the sum of the digits of x, is divisible by 3.

$\Longleftarrow$ (This implication symbol means we are proving the reverse direction. "If the sum of the digits of x is divisible by 3, then x is divisible by 3.")

Since the sum of the digits of x is divisible by 3, $a + b = 3k$ for some integer k.

$$\begin{aligned} a + b &= 3k && \\ 10a + 10b &= 30k && \text{(multiply both sides by 10)} \\ 10a + b &= 30k - 9b && \text{(subtract } 9b \text{ from both sides)} \\ 10a + b &= 3(10k - 3b) && \text{(common factor a 3)} \\ 10a + b &= 3m && \text{(where } m = 10k - 3b\text{)} \\ x &= 3m && \text{(since } x = 10a + b\text{)} \end{aligned}$$

Therefore, x is divisible by 3.

□

By proving both directions, we proved the if and only if statement.

This divisiblity rule is true for numbers with as many digits as you like!

Exercise 10
Determine if the using following number are divisible by 3 using the rule from Example 10.

(a) Is 84 divisible by 3?

(b) Is 186 divisible by 3?

(c) Is 4592 divisible by 3?

(d) Write down a five-digit number that is divisible by 3.

Stop and Think

What other divisibility rules can you think of?

This concludes the lesson, but it is just the beginning of proofs! There's an incredible number of theorems and proofs that help mathematicians solve problems.

There are also *unsolved* problems which people are currently working to prove. The Collatz Conjecture is one of these problems.

Watch this video if you're curious about this unsolved problem! https://www.youtube.com/watch?v=094y1Z2wpJg

Summary

Today was all about proving that a statement is true **for all cases** or proving a statement is false with a counter-example.

We saw three types of statements:

- Implication (if, then statement)
- Converse (reverse order of an implication)
- If and Only If (implication and converse in one statement)

We also saw a few proof strategies:

- To prove an implication or its converse, assume the **hypothesis** is true and prove the **conclusion** is true.
- To prove an if and only of statement, prove both the implication and the converse!
- To disprove any statement, find a **counter-example** such that the hypothesis is true but the conclusion is false.

Finally, there were some common themes for the proof questions that we saw:

- Even/odd questions
  - Any even number can be written as $2k$ for some integer k so to show a number x is even, show $x = 2k$ for some integer k.
  - Any odd number can be written as $2k + 1$ for some integer k so to show a number x is odd, show $x = 2k + 1$ for some integer k.
- Divisibility questions
  - To show a number x is divisible by 6, for example, show that $x = 6k$ for some integer k.